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Propositional Systems and Measurements—II

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Abstract

A direct approach to propositional systems alternative to that of Jauch and Piron has been given in a preceding paper. The most difficult point is the foundation of the covering law. Independently of the latter holding true we consider here questions defined by a measurement process which we describe in the propositional system of the apparatus coupled to the quantum object.

I

A direct approach to propositional systems alternative to that of Jauch and Piron (Jauch & Piron, 1969) has been given in a previous paper (Hellwig & Krausser, 1974), henceforth referred to as PSMI. It is based on a set $\mathscr{S}$ of preparation procedures and a set $\mathscr{Q}$ of questions. Classes of equivalent questions are introduced in the same manner as by Jauch and Piron to define propositions. The set of Propositions, denoted by $[\mathscr{Q}]$, is partially ordered in a natural way such that $([\mathscr{Q}], \leq)$ is automatically a complete lattice. We have then departed from Jauch and Piron's way of reasoning to introduce orthocomplementarity, weak modularity and atomicity, step by step, through postulates on $\mathscr{S}$ or $\mathscr{Q}$.

We have said that a physical system has property $([\alpha])$ iff the proposition $[\alpha]$, $\alpha \in \mathscr{Q}$, holds true. Physical systems on which $([\alpha])$ is present can be produced by certain preparation procedures which impose $[\alpha]$ to hold true. We have given an operational definition for $([\alpha])$ to be absent on physical systems. There are propositions $[\nu\gamma]$ such that $([\alpha])$ is absent if $[\nu\gamma]$ holds true, but there need not be a proposition which necessarily holds true if $([\alpha])$ is absent. In Postulate 1 we have required such a proposition to exist and called it $\varphi([\alpha])$. So a mapping $\varphi : [\mathscr{Q}] \to [\mathscr{Q}]$ has been defined. We found φ to be an orthocomplementation iff $\varphi([\mathscr{Q}]) = [\mathscr{Q}]$, i.e. iff φ is subjective. This has been required in Postulate 2. Weak modularity now turned out to be equivalent to the statement: If $[\alpha] \leq [\beta]$ and $[\alpha] \neq [\beta]$ then there is at least one preparation procedure for physical systems in which ($[\beta]$) is present but ($[\alpha]$) is absent. Postulate 3 is just this statement. Considering the sets of propositions imposed to be true, or, otherwise stated, sets of properties imposed to be present by certain preparation procedures, we have introduced equivalence classes of sets of preparation procedures. These classes have been partially ordered in a natural way: The more properties imposed the better is the class of preparation procedures. Postulate 4, a requirement that some kind of the best preparation procedures exist, implies atomicity† of the propositional system.

The covering law has been introduced by Jauch and Piron through a postulate which requires ideal measurements of the first kind to exist for any proposition. It seems questionable to us whether this postulate can be fulfilled in nature. Instead, we have introduced and discussed a superposition postulate, which seems to be of some advantage. Unfortunately the latter postulate does not make any statement in classical physics, hence it is far from being obvious on the level of everyday experience.

If the covering law generally holds true for propositional systems, quantum mechanics can be described in Hilbert space. In Hilbert space measurement processes which bring information about quantum objects into the classical level of observation can be formulated. Can this also be done when the covering law does not hold? To answer this problem one has to describe measurement processes independently from Hilbert space representation of the propositional system for the quantum object, only assuming Postulates 1 through 4 to hold true for the latter.

In the following we attempt to give a formulation of measuring processes without assuming the covering law to hold true for the propositional system of the quantum object. To any question belongs a classical system, the apparatus, from which the outcome yes or no is to be read by the experimenter. In the simplest case the quantum object is coupled with the apparatus to define another system, composed of both. Initially, both systems are free, i.e. no interaction takes place between the components, and both are prepared independently in prescribed ways. An interaction may then happen. Finally the experimenter observes the outcome of a certain question for the compound system.

In Section II we introduce a postulate which relates the propositional system of the compound system to those of the components, quantum object and apparatus. We do not specify the interaction process which may happen between the components, but note that the question, we denote it by μ_{12}, defining final observation is contained in a proposition of the compound system. In Section III we connect the propositions $[\epsilon_1]$ and $[\nu\epsilon_1]$ defined by the measurement process, with the preparation procedure of the apparatus and the proposition $[\mu_{12}]$, finally to be observed. From Postulates 1 and 2 of PSMI it follows that any proposition must contain at least one question ϵ_1 which fulfills $[\nu\epsilon_1] = \varphi[\epsilon_1]$. A set of necessary and sufficient conditions for this special case is derived in Section IV. We compare briefly the Hilbert space

† Atomicity is used in the sense: Every proposition can be represented as a joining of atoms.

formulation with that given here. Finally, in Section V, we show that commensurability of two propositions implies compatibility.

II

In the description of a measurement process we have to consider the physical system composed of a quantum object and apparatus. The propositional systems of quantum object and apparatus will be denoted by $([\mathscr{Q}_1], \leq, \varphi)$ and $([\mathscr{Q}_2], \leq, \varphi)$, respectively, and that of the compound system by $([\mathscr{Q}_{12}], \leq, \varphi)$. We will use the same symbols for the orderings and lattice operations in the different propositional systems. Questions, propositions and further notations will be labelled with lower indices 1, 2, 12 if they refer to the three propositional systems, respectively.

We must know how propositions of the component systems, quantum object and apparatus, are related to propositions of the compound system. Such assumptions have been formulated by Jauch (Jauch, 1968). Close investigations in connection with Ludwig's axiomatic approach to quantum mechanics have been given by Hartkämper (Hartkämper, 1968) and Neumann (Neumann, 1968). We content ourselves with some basic ideas for a partial motivation of the following, Postulate 5, which we adopt for our purpose.

We assume that, at least if the component systems are sufficiently far away from each other, they can be treated as independent (or 'free') physical systems. Preparation procedures on the component systems can then be carried through independently from each other arbitrarily. Any proposition $[\alpha_1]$ or $[\alpha_2]$ can be imposed to hold true for object or apparatus by at least one preparation procedure $s_1, s_1 \in \tilde{\mathscr{S}}_1$, or $s_2, s_2 \in \tilde{\mathscr{S}}_2$, respectively. But any set of preparation procedures $\mathfrak{s}_j (j = 1, 2)$, $\mathfrak{s}_j \subseteq \tilde{\mathscr{S}}_j$, is also a set of preparation procedures for the compound system. We know from PSMI that $\mathfrak{s}_j$ determines a minimal proposition $a([\mathfrak{s}_j])$ in $[\mathscr{Q}_j]$ which is imposed by any procedure $s_j, s_j \in \mathfrak{s}_j$. If the compound system is considered, there must be such minimal proposition in $[\mathscr{Q}_{12}]$ which we denote by $\Theta_j(a([\mathfrak{s}_j]))$. We require

Postulate 5.1. For any component system there is an injection

$$\Theta_j\colon [\mathscr{Q}_j] \to [\mathscr{Q}_{12}] \qquad (j = 1, 2)$$

such that

(i) $$\Theta_j\Big(\bigwedge_{[\alpha_j] \in \mathscr{K}} [\alpha_j]\Big) = \bigwedge_{[\alpha_j] \in \mathscr{K}} \Theta_j([\alpha_j])$$

for arbitrary $\mathscr{K}$, $\mathscr{K} \subseteq [\mathscr{Q}_j]$

(ii) $$\Theta_j([\phi_j]) = [\phi_{12}]$$

and

(iii) $$\varphi \circ \Theta_j = \Theta_j \circ \varphi$$

The following conclusions are immediate.

Lemma 1.

(i)′ $$\Theta_j(\bigvee_{[\alpha_j]\in\mathscr{K}}[\alpha_j]) = \bigvee_{[\alpha_j]\in\mathscr{K}}\Theta_j([\alpha_j])$$

(ii)′ $$\Theta_j([I_j]) = [I_{12}]$$

As long as the component systems are separated from each other, measurements on different components can be carried through independently from each other. So we assume that $\Theta_1([\alpha_1])$ is always compatible with $\Theta_2([\beta_2])$ (in symbols $\Theta_1([\alpha_1]) \leftrightarrow \Theta_2([\beta_2])$). Moreover no proposition $\Theta_1([\alpha_1])$ can imply or can be implied by any $\Theta_2([\beta_2])$, if $[\alpha_1] \neq [\phi_1][I_1]$, and $[\beta_2] \neq [\phi_2], [I_2]$.

Postulate 5.2. Let $[\alpha_j], [\beta_j] \in [\mathscr{Q}_j], (j = 1, 2)$.

(iv) If $$[\alpha_j], [\beta_j] \neq [\phi_j], [I_j]$$

then

$$\Theta_j([\alpha_j]) \leq \Theta_k([\beta_k])$$

implies $j = k$.

(v) $$\Theta_1([\alpha_1]) \leftrightarrow \Theta_2([\beta_2])$$

Maximal boolean sublattices $\mathscr{B}_j$ of $[\mathscr{Q}_j]$ correspond, in the usual terminology, to complete systems of observables for the components, respectively. We assume finally

Postulate 5.3. Let us be given any two maximal boolean sublattices $\mathscr{B}_j$, $\mathscr{B}_j \subseteq [\mathscr{Q}_j]$ $(j = 1, 2)$. Then the completion by cuts of the boolean sublattice $\mathscr{B}_{12}$ generated by $\Theta_1(\mathscr{B}_1) \cup \Theta_2(\mathscr{B}_2)$ is maximal in $[\mathscr{Q}_{12}]$.

With help of the latter postulate we prove

Lemma 2. Let $e_j (j = 1, 2)$ denote atoms in $[\mathscr{Q}_j]$. Then $\Theta_1(e_1) \wedge \Theta_2(e_2)$ is an atom in $[\mathscr{Q}_{12}]$.

Proof. Let us be given any boolean lattice $\mathscr{B}$, and $e\in\mathscr{B}$, $e \neq [\phi]$. One can easily check that e is an atom in $\mathscr{B}$ iff for any $a, a\in\mathscr{B}$, $e \nleq a$ implies $a \leq \varphi e$.

Now arbitrarily consider two maximal boolean sublattices $\mathscr{B}_j$ of $[\mathscr{Q}_j]$ $(j = 1, 2)$ which contain the given atoms e_j of $[\mathscr{Q}_j]$, respectively. For any $a_j\in\mathscr{B}_j$, $\Theta_j(e_j) \nleq \Theta_j(a_j)$ implies $\Theta_j(a_j) \leq \Theta_j(\varphi e_j) = \varphi\Theta_j(e_j)$. Denote by $\mathscr{B}_{12}$ the boolean sublattice which arises as the completion by cuts of the boolean lattice generated by $\Theta_1(\mathscr{B}_1) \cup \Theta_2(\mathscr{B}_2)$ in $[\mathscr{Q}_{12}]$. Any $a_{12}\in\mathscr{B}_{12}$ can be written in conjunctive normal form, i.e., if we drop indexing sets, in the form $a_{12} = \wedge b_{12}$ with $b_{12} = \vee c_{12}$ and $c_{12}\in\Theta_1(\mathscr{B}_1) \cup \Theta_2(\mathscr{B}_2)$. From Lemma 1 we infer $b_{12} = \Theta_1(a_1) \vee \Theta_2(a_2)$ with $a_1 \in\mathscr{B}_1$ and $a_2\in\mathscr{B}_2$. So a_{12} finally can be written in the form $a_{12} = \bigwedge_{\mathscr{K}_1 \times \mathscr{K}_2}(\Theta_1(a_1) \vee \Theta_2(a_2))$ with a suitable subset $\mathscr{K}_1 \times \mathscr{K}_2$ of $\mathscr{B}_1 \times \mathscr{B}_2$.

Consider $b_{12} = \Theta_1(a_1) \vee \Theta_2(a_2)$. $\Theta_1(e_1) \wedge \Theta_2(e_2) \nleq b_{12}$ implies $e_1 \nleq a_1$, and $e_2 \nleq a_2$. Hence $a_1 \leq \varphi e_1$ and $a_2 \leq \varphi e_2$. So $b_{12} \leq \varphi\Theta_1(e_1) \vee \varphi\Theta_2(e_2)$ $= \varphi(\Theta_1(e_1) \wedge \Theta_2(e_2))$.

Consider now $a_{12} = \bigwedge_{\mathscr{K}_1 \times \mathscr{K}_2} (\Theta_1(a_1) \vee \Theta_2(a_2))$. $\Theta_1(e_1) \wedge \Theta_2(e_2) \nleq a_{12}$ implies $\Theta_1(e_1) \wedge \Theta_2(e_2) \nleq \Theta_1(a_1) \vee \Theta_2(a_2)$ for some $(a_1, a_2) \in \mathscr{K}_1 \times \mathscr{K}_2$. So $\Theta_1(a_1) \vee \Theta_2(a_2) \leq \varphi(\Theta_1(e_1) \wedge \Theta_2(e_2))$ for any such (a_1, a_2). So $a_{12} \leq \varphi(\Theta_1(e_1) \wedge \Theta_2(e_2))$ if $\Theta_1(e_1) \wedge \Theta_2(e_2) \nleq a_{12}$ holds true for any $a_{12} \in \mathscr{B}_{12}$.

We show that $\Theta_1(e_1) \wedge \Theta_2(e_2) \neq [\phi_{12}]$. Assume in contrary $\varphi(\Theta_1(e_1) \wedge \Theta_2(e_2)) = \varphi\Theta_1(e_1) \vee \varphi\Theta_2(e_2) = [I_{12}]$ to hold true. Then $\Theta_1(e_1) \wedge \varphi\Theta_2(e_2) = \Theta_1(e_1)$, i.e. $\Theta_2(\varphi e_2)) \geq \Theta_1(e_1)$ which gives, by Postulate 5 (iv), $1 = 2$, a contradiction.

By the argument given at the beginning of this proof we have shown $\Theta_1(e_1) \wedge \Theta_2(e_2)$ to be an atom in $\mathscr{B}_{12}$. We have to show that it is also an atom in $[\mathscr{Q}_{12}]$. Let $l_{12} \in [\mathscr{Q}_{12}], e_{12} \leq \Theta_1(e_1) \wedge \Theta_2(e_2)$, then for any a_{12}, $a_{12} \in \mathscr{B}_{12}$, we have $e_{12} \leq a_{12}$ or $a_{12} \leq \varphi e_{12}$. So $e_{12} \leftrightarrow a_{12}$. By Postulate 5.3, the sublattice $\mathscr{B}_{12}$ is maximal boolean in $[\mathscr{Q}_{12}]$, hence $e_{12} \in \mathscr{B}_{12}$. Assume $e_{12} \neq [\phi_{12}]$, then $e_{12} \nleq \varphi(\Theta_1(e_1) \wedge \Theta_2(e_2))$ gives $\Theta_1(e_1) \wedge \Theta_2(e_2) \leq e_{12}$. So $e_{12} = \Theta_1(e_1) \wedge \Theta_2(e_2)$. This completes the proof of Lemma 2.

Lemma 3. Let $[\alpha_1], [\beta_1] \in [\mathscr{Q}_1]$, and $[\gamma_2] \in [\mathscr{Q}_2]$, then

$$[\alpha_1] \leq [\beta_1] \quad \text{iff} \quad \Theta_1([\alpha_1]) \wedge \Theta_2([\gamma_2]) \leq \Theta_1([\beta_1]) \wedge \Theta_2([\gamma_2])$$

Proof. Necessity is trivial. To show sufficiency assume

$$\Theta_1([\alpha_1]) \wedge \Theta_2([\gamma_2]) \leq \Theta_1([\beta_1]) \wedge \Theta_2([\gamma_2])$$

but, contrary to the statement, $[\alpha_1] \wedge [\beta_1] \neq [\alpha_1]$. Then, by Postulate 3 of PSMI, there is a $r_1, r_1 \in \tilde{\mathscr{S}}_1$, such that $a_1([r_1]) \leq [\alpha_1]$ and $a_1([r_1]) \leq \varphi([\alpha_1] \wedge [\beta_1])$.†

The latter relation implies that

$$[\phi_{12}] \neq \Theta_1 \circ a_1([r_1]) \wedge \Theta_2([\gamma_2]) \leq \varphi\Theta_1([\alpha_1] \wedge [\beta_1]) \wedge \Theta_2([\gamma_2])$$

So $\Theta_1 \circ a_1([r_1]) \wedge \Theta_2([\gamma_2])$ is nontrivial and precedes the orthocomplement of $\Theta_1([\alpha_1] \wedge [\beta_1]) \wedge \Theta_2([\gamma_2])$. Hence

$$\Theta_1([\alpha_1] \wedge [\beta_1]) \wedge \Theta_2([\gamma_2]) \neq \Theta_1(([\alpha_1] \wedge [\beta_1]) \vee a_1([r_1])) \wedge \Theta_2([\gamma_2])$$

Moreover, from $a_1([r_1]) \leq [\alpha_1]$ we have

$$\Theta_1(([\alpha_1] \wedge [\beta_1]) \vee a_1([r_1])) \wedge \Theta_2([\gamma_2]) \leq \Theta_1([\alpha_1]) \wedge \Theta_2([\gamma_2])$$

Combining both results we obtain

$$\Theta_1([\alpha_1] \wedge [\beta_1]) \wedge \Theta_2([\gamma_2]) \neq \Theta_1([\alpha_1]) \wedge \Theta_2([\gamma_2])$$

which contradicts the assumption.

III

We come now to our main problem, the description of measurement processes. We require Postulates 1 through 5 to hold true. The covering law need not be true in any of the propositional systems in consideration.

† We have written $[r_1]$ instead of $[\{r_1\}]$ to make notations not too complicated.

Recall that we are considering a series of experiments which are carried through in accordance with a certain prescription which we have denoted by (s_1, ϵ_1), $s_1 \in \tilde{\mathscr{S}}_1$, $\epsilon_1 \in \mathscr{Q}_1$. Note that all our argumentation rests on the assumption of homogeneity of time. For any single experiment we choose as zero point of time the preparation of the quantum object. The latter is given by the moment at which a certain preparative manipulation, prescribed by s_1 has been carried out. The measurement prescription ϵ_1 fixes among other things an admissible set $\mathfrak{r}_2$ of preparation procedures for the apparatus in such geometrical and temporal relation to s_1, that apparatus and object are prepared independently from each other. Finally, ϵ_1 prescribes a certain observation which has to be done at a certain time on the compound system. In our case this is a question μ_{12}, $\mu_{12} \in \mathscr{Q}_{12}$. The result of this observation defines the outcome, either yes or no, for ϵ_1 and concludes the measurement process.

In the usual description of measurement processes in Hilbert space a unitary operator describes the temporal development of the compound system due to the interaction of quantum object and apparatus. Here we do not analyse this interaction process. We only use the very meaning of preparation procedures, questions, and the coupling of physical systems. The independent preparation of quantum object and apparatus imposes a certain minimal proposition $[\alpha_{12}]$, $[\alpha_{12}] \in [\mathscr{Q}_{12}]$, to hold true for the compound system. Minimality means that any further true proposition in $[\mathscr{Q}_{12}]$ is implied by $[\alpha_{12}]$. If we determine $[\alpha_{12}]$ for a given set $\mathfrak{s}_1$ of preparation procedures in $\mathscr{S}_1$, then $[\alpha_{12}] \leq [\mu_{12}]$ will be a criterion for $\mathfrak{s}_1 \subseteq \mathscr{O}(\epsilon_1)$. We proceed in this way.

The set

$$t(\mathfrak{r}) = \{[\alpha] \,|\, a([\mathfrak{r}]) \leq [\alpha]\} \subseteq [\mathscr{Q}]$$

of all propositions imposed to be true for any preparation procedure in a given set $\mathfrak{r}$ has been introduced in Chapter V of PSMI. Note that $a([\mathfrak{r}])$ and, consequently, $t(\mathfrak{r})$ only depend on the class $[\mathfrak{r}]$ of equivalent sets of preparation procedures and not on the special representing set $\mathfrak{r}$. We have written also

$$[\mathfrak{s}] < [\mathfrak{r}] \quad \text{for} \quad a([\mathfrak{s}]) \leq a([\mathfrak{r}])$$

The meaning of the latter partial ordering is, that all s, $s \in \mathfrak{s}$, impose all propositions to be true which r, $r \in \mathfrak{r}$, imposes, but there may be some more. Let $\mathscr{T}([\mathfrak{r}])$ denote the ensemble of pure sets or preparation procedures $\check{\mathfrak{r}}$ with $[\check{\mathfrak{r}}] < [\mathfrak{r}]$.

We state

Lemma 4. $a([\mathfrak{r}]) = \bigvee_{\check{\mathfrak{r}} \in \mathscr{T}([\mathfrak{r}])} a([\check{\mathfrak{r}}])$

Proof. As $a([\mathfrak{r}]) \geq a([\check{\mathfrak{r}}])$, we have immediately

$$a([\mathfrak{r}]) \geq \bigvee_{\check{\mathfrak{r}} \in \mathscr{T}([\check{\mathfrak{r}}])} a([\check{\mathfrak{r}}])$$

Assume, contrary to the statement, inequality to hold, then, by Postulate 3, there is a preparation procedure r with $a([r]) \leq a([\mathfrak{r}])$ and $a([r]) \leq \varphi(\bigvee_{\check{\mathfrak{r}} \in \mathscr{T}([\mathfrak{r}])} a([\check{\mathfrak{r}}]))$. Moreover, by Postulate 4, there is a pure set $\check{\mathfrak{s}}$, such that $a([\check{\mathfrak{s}}]) \leq a([r])$. This implies $\check{\mathfrak{s}} \in \mathscr{T}([\mathfrak{r}])$, so $a([\check{\mathfrak{s}}]) \leq \bigvee_{\check{\mathfrak{r}} \in \mathscr{T}([\mathfrak{r}])} a([\check{\mathfrak{r}}])$, which contradicts $a([\check{\mathfrak{s}}]) \leq \varphi(\bigvee_{\check{\mathfrak{r}} \in \mathscr{T}([\mathfrak{r}])} a([\check{\mathfrak{r}}]))$. This completes the proof.

With this in mind, we return to the measurement of ϵ_1. Quantum object and apparatus have to be prepared independently from each other, the latter by a procedure in $\mathfrak{r}_2$. Let the quantum object be prepared by a procedure in $\mathfrak{s}_1$. As already mentioned in Section II, by this a certain set of preparation procedures for the compound system is defined, which we denote by $(\mathfrak{s}_1, \mathfrak{r}_2)$. We note that, obviously, $[\check{\mathfrak{s}}_1] < [\mathfrak{s}_1]$ and $[\check{\mathfrak{r}}_2] < [\mathfrak{r}_2]$ imply $[(\check{\mathfrak{s}}_1, \check{\mathfrak{r}}_2)] < [(\mathfrak{s}_1, \mathfrak{r}_2)]$. We are seeking the minimal $[\alpha_{12}]$ in $[\mathscr{Q}_{12}]$ which is imposed by any procedure in $(\mathfrak{s}_1, \mathfrak{r}_2)$, i.e. we have to seek $a_{12}([(\mathfrak{s}_1, \mathfrak{r}_2)])$. From Section II we already know $\Theta_1(a_1([\mathfrak{s}_1]))$, and $\Theta_2(a_2([\mathfrak{r}_2]))$ to hold true in that case. So we have immediately

$$a_{12}([(\mathfrak{s}_1, \mathfrak{r}_2)]) \leq \Theta_1 \circ a_1([\mathfrak{s}_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

Let now $\mathscr{T}_1([\mathfrak{s}_1])$ and $\mathscr{T}_2([\mathfrak{r}_2])$ be defined analogously to $\mathscr{T}([\mathfrak{r}])$ in Lemma 4. We have by the same argument

$$a_{12}([(\check{\mathfrak{s}}_1, \check{\mathfrak{r}}_2)]) = \Theta_1 \circ a_1([\check{\mathfrak{s}}_1]) \wedge \Theta_2 \circ a_2([\check{\mathfrak{r}}_2])$$

for any $\check{\mathfrak{s}}_1 \in \mathscr{T}_1([\mathfrak{s}_1])$ and $\check{\mathfrak{r}}_2 \in \mathscr{T}_2([\mathfrak{r}_2])$, because the right-hand side is an atom in $[\mathscr{Q}_{12}]$, by Lemma 2. From $[(\check{\mathfrak{s}}_1, \check{\mathfrak{r}}_2)] < [(\mathfrak{s}_1, \mathfrak{r}_2)]$ we have, in addition,

$$\Theta_1 \circ a_1([\check{\mathfrak{s}}_1]) \wedge \Theta_2 \circ a_2([\check{\mathfrak{r}}_2]) = a_{12}([(\check{\mathfrak{s}}_1, \check{\mathfrak{r}}_2)]) \leq a_{12}([(\mathfrak{s}_1, \mathfrak{r}_2)])$$

With the help of Lemma 1(i)′, and Postulate 5(v) we conclude

$$\Theta_1(\bigvee_{\check{\mathfrak{s}}_1 \in \mathscr{T}([\mathfrak{s}_1])} a_1([\check{\mathfrak{s}}_1])) \wedge \Theta_2(\bigvee_{\check{\mathfrak{r}}_2 \in \mathscr{T}([\mathfrak{r}_2])} a_2([\check{\mathfrak{r}}_2])) \leq a_{12}([(\mathfrak{s}_1, \mathfrak{r}_2)])$$

or, if we apply Lemma 4,

$$\Theta_1 \circ a_1([\mathfrak{s}_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq a_{12}([(\mathfrak{s}_1, \mathfrak{r}_2)])$$

We have proven

Theorem 1. The set of all propositions imposed as true for any preparation procedure in $(\mathfrak{s}_1, \mathfrak{r}_2)$ is given by

$$t_{12}((\mathfrak{s}_1, \mathfrak{r}_2)) = \{[\alpha_{12}] \mid \Theta_1 \circ a_1([\mathfrak{s}_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\alpha_{12}]\}$$

The criterion for $\mathfrak{s}_1 \subseteq \mathscr{O}(\epsilon_1)$ is now

$$\Theta_1 \circ a_1([\mathfrak{s}_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\mu_{12}]$$

In PSMI we saw that the mapping a: $[\mathscr{P}(\check{\mathscr{S}})] \to [\mathscr{Q}] \setminus \{[\phi]\}$ is bijective. We have, obviously, for $[\alpha] \neq [\phi]$ that $[\alpha] = a([\mathscr{O}(\alpha)])$. So we may restate the criterion as

$$[\alpha_1] \leq [\epsilon_1] \quad \text{iff} \quad \Theta_1([\alpha_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\mu_{12}]$$

Recall that $\nu\epsilon_1$ is defined to have outcome yes iff the outcome of ϵ_1 is no. So we may add

$$[\beta_1] \leq [\nu\epsilon_1] \quad \text{iff} \quad \Theta_1([\beta_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\nu\mu_{12}]$$

We sum up in

Theorem 2. Let ϵ_1 be the question defined by a measurement process with $\mathfrak{r}_2$ as the set of admissible preparation procedures of the apparatus and μ_{12} the question finally to be observed on the compound system. Then

$$\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\mu_{12}]$$

$$\Theta_1([\nu\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\nu\mu_{12}]$$

Moreover, $[\epsilon_1]$ and $[\nu\epsilon_1]$ are determined by

$$[\epsilon_1] = \text{l.u.b.}\{[\alpha_1] \mid \Theta_1([\alpha_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\mu_{12}]\}$$

and

$$[\nu\epsilon_1] = \text{l.u.b.}\{[\beta_1] \mid \Theta_1([\beta_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\nu\mu_{12}]\}$$

From the definition of $[\epsilon_1]$ in PSMI we know that $[\nu\epsilon_1] \leq \varphi[\epsilon_1]$ must hold true. We note that this is in accord with theorem 2. We have

$$\Theta_1([\nu\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\nu\mu_{12}]$$

$$\leq \varphi[\mu_{12}] \leq \Theta_1(\varphi[\epsilon_1]) \vee \varphi\Theta_2 \circ a_2([\mathfrak{r}_2])$$

If we meet with $\Theta_2 \circ a_2([\mathfrak{r}_2])$ and use postulate 5(v), this gives

$$\Theta_1([\nu\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq \Theta_1(\varphi[\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

which, by Lemma 3, is equivalent to $[\nu\epsilon_1] \leq \varphi[\epsilon_1]$.

The relations stated in Theorem 2 can be written equivalently in the form

$$\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

$$\Theta_1([\nu\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\nu\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

The compound system is conditioned by truth of $\Theta_2 \circ a_2([\mathfrak{r}_2])$. So any further proposition $[\alpha_{12}]$ which holds true for the compound systems is in conjunction with $\Theta_2 \circ a_2([\mathfrak{r}_2])$. Only such conjunctions are related in Theorem 2. As a sublattice, the segment $[[\phi_{12}], \Theta_2 \circ a_2([\mathfrak{r}_2])]$, i.e. the lattice propositions $[\beta_{12}]$, $[\beta_{12}] \leq \Theta_2 \circ a_2([\mathfrak{r}_2])$, can be endowed with the canonical relative orthocomplement φ_r which is defined by

$$\varphi_r[\beta_{12}] = \varphi[\beta_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

It is well known that

$$([[\phi_{12}], \Theta_2 \circ a_2([\mathfrak{r}_2])], \leq, \varphi_r)$$

is weakly modular (Piron, 1964, Appendice II). We may consider it as the propositional system of the compound system conditioned by the preparation of the apparatus.

IV

For a question $\alpha_1, \alpha_1 \in \mathscr{Q}_1$, $[\nu\alpha_1] \leq \varphi[\alpha_1]$ holds true in general. In PSMI we have required Postulates 1 and 2, i.e., that any proposition contains at least one question ϵ_1, such that $[\nu\epsilon_1] = \varphi[\epsilon_1]$. We shall now assume ϵ_1 to be defined by a measurement process as introduced above and state a necessary and sufficient condition for this to hold true.

We assume $[\nu\mu_{12}] = \varphi[\mu_{12}]$ to hold true for the compound system. This is obvious for classical observation. We have dropped this assumption in Theorem 2 because it does not enter into the proof and makes possible the study of v. Neumann chains.

Theorem 3. In addtion to the assumptions of Theorem 2 let

$$[\nu\mu_{12}] = \varphi[\mu_{12}]$$

hold true. Then

$$[\nu\epsilon_1] = \varphi[\epsilon_1]$$

holds true if and only if the three conditions

(i) $$\Theta_2 \circ a_2([\mathfrak{r}_2]) \leftrightarrow [\mu_{12}]$$

(ii) $$\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = [\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

(iii) $$\Theta_1([\nu\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = \varphi[\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

are fulfilled.

Proof. Let φ_r denote the canonical relative orthocomplement with respect to the segment $[[\phi_{12}], \Theta_2 \circ a_2([\mathfrak{r}_2])]$ of $[\mathscr{Q}_{12}]$. For $[\alpha_{12}] \in [\mathscr{Q}_{12}]$ and $[\alpha_{12}] \leftrightarrow \Theta_2 \circ a_2([\mathfrak{r}_2])$ we have always

$$\begin{aligned}\varphi_r([\alpha_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]))& \\ &= \varphi([\alpha_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \\ &= (\varphi[\alpha_{12}] \vee \varphi\Theta_2 \circ a_2([\mathfrak{r}_2])) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \\ &= \varphi[\alpha_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])\end{aligned}$$

Note especially that

$$\varphi_r(\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])) = \Theta_1(\varphi[\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

holds true.

Now assume $[\nu\epsilon_1] = \varphi[\epsilon_1]$. From Theorem 2 we then have

$$\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) \leq [\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

and

$$\varphi_r(\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])) \leq (\varphi[\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]))$$

These relations imply (i), (ii), and (iii) of the theorem. Firstly, we have

$$([\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])) \vee (\varphi[\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])) = \Theta_2 \circ a_2([\mathfrak{r}_2])$$

which implies (i). From (i) we conclude

$$\varphi_r([\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])) = (\varphi[\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]))$$

Hence, the second relation writes

$$\varphi_r(\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])) \leq \varphi_r([\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]))$$

which is equivalent to

$$(\Theta_1[\epsilon_1] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])) \geq ([\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]))$$

This proves (ii), and, in consequence, (iii), because φ_r is an orthocomplementation. So the conditions are proved to be necessary.

We now show sufficiency. From (i) follows

$$\varphi_r([\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])) = \varphi[\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

Conditions (ii) and (iii) now give

$$\Theta_1([\nu\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = \varphi_r(\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]))$$
$$= \Theta_1(\varphi[\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

Application of Lemma 2 leads to $[\nu\epsilon_1] = \varphi[\epsilon_1]$. This completes the proof of Theorem 3.

We briefly compare the foregoing with the usual description of the measurement process in Hilbert space. Let $\mathfrak{H}_1$, $\mathfrak{H}_2$, and $\mathfrak{H}_1 \otimes \mathfrak{H}_2$ denote the state spaces of quantum object, apparatus, and compound system, respectively. Let W_2 denote the density operator of the apparatus which acts in $\mathfrak{H}_2$. We assume that the question finally to be observed fulfills $[\nu\mu_{12}] = \varphi[\mu_{12}]$. It is then represented by a projection M_{12} in $\mathfrak{H}_{12}$. We note that, usually, M_{12} is written as $M_{12} = S^{+}Q_{12}S$, where S denotes the unitarian representing time translation of the compound system from initial preparation to final observation in the Heisenberg picture. Let W_1, acting in $\mathfrak{H}_1$, denote the density operator of the quantum object, then the probability for the outcome yes is

$$\mathrm{tr}_{12}((W_1 \otimes W_2)M_{12}) = \mathrm{tr}_1(F_1 W_1) \qquad (*)$$

tr_{12} denotes the trace in $\mathfrak{H}_1 \otimes \mathfrak{H}_2$, tr_1 denotes that in $\mathfrak{H}_1$. The hermitean F_1, arising on the right-hand side of the equation, is uniquely determined by W_2 and M_{12}. The projection E_1^1 on the eigenspace to eigenvalue 1 of F_1 represents the proposition $[\epsilon_1]$, the projection E_1^0 corresponding to eigenvalue 0 of F_1 represents $[\nu\epsilon_1]$. In general $E_1^0 \leq 1 - E_1^1$ holds true, corresponding to $[\nu\epsilon_1] \leq \varphi[\epsilon_1]$.

Obviously, $a_2([\mathfrak{r}_2])$ corresponds to the projection on the support of W_2 in $\mathfrak{H}_2$, which we denote by $A_2(W_2)$. Theorem 2 then states

$$E_1^1 \otimes A_2(W_2) \leq M_{12}$$

$$E_1^0 \otimes A_2(W_2) \leq 1 - M_{12}.$$

It is an easy matter to derive these equations and the analogue of the second statement of Theorem 2 directly from (*). Less trivial is the statement of Theorem 3 if rewritten in terms of Hilbert space quantum mechanics: F_1 is a projection, i.e. $E_1^0 = 1 - E_1^1$, if and only if,

(i) $$(1 \otimes A_2(W_2))M_{12} - M_{12}(1 \otimes A_2(W_2)) = 0$$

(ii) $$E_1^1 \otimes A_2(W_2) = M_{12}(1 \otimes A_2(W_2))$$

(iii) $$E_1^0 \otimes A_2(W_2) = (1 - M_{12})(1 \otimes A_2(W_2))$$

V

There are measurement processes in the course of which, together with $[\mu_{12}]$, another proposition, say $[\rho_{12}]$, can finally be observed. We have to presuppose $[\mu_{12}] \leftrightarrow [\rho_{12}]$, which is clear, if final observations are classical. Two questions, we denote them by ϵ_1 and δ_1, are then defined by one measurement process due to observation of $[\mu_{12}]$ or $[\rho_{12}]$, respectively. In the terminology of G. Ludwig ϵ_1 is then called coexistent with δ_1. If, moreover, $[\nu\epsilon_1] = \varphi[\epsilon_1]$ and $[\nu\delta_1] = \varphi[\delta_1]$, we call $[\epsilon_1]$ commensurable with $[\delta_1]$. Any proposition of the boolean sublattice of $[\mathscr{Q}_{12}]$ generated by $\{[\mu_{12}], [\rho_{12}]\}$ defines a question on the quantum object, and observation of both $[\mu_{12}]$ and $[\rho_{12}]$ fixes yes or no for each. One expects the sublattice of $[\mathscr{Q}_1]$, which is generated by $\{[\epsilon_1], [\delta_1]\}$, to also be boolean, or, in other words, $[\epsilon_1] \leftrightarrow [\delta_1]$.

Theorem 4. Let $[\epsilon_1]$ and $[\delta_1]$ be commensurable and defined as above by final observation of $[\mu_{12}]$ or $[\rho_{12}]$, $[\mu_{12}] \leftrightarrow [\rho_{12}]$, on the compound system, respectively. Then $[\epsilon_1]$ and $[\delta_1]$ are compatible and, moreover, final observation of any proposition of the boolean sublattice of $[\mathscr{Q}_{12}]$ generated by $\{[\mu_{12}], [\rho_{12}]\}$ defines a question α_1 which fulfills $[\nu\alpha_1] = \varphi[\alpha_1]$.

Proof. From assumptions we have besides $[\mu_{12}] \leftrightarrow [\rho_{12}]$ also $\Theta_2 \circ a_2([\mathfrak{r}_2]) \leftrightarrow [\mu_{12}]$, and $\Theta_2 \circ a_2([\mathfrak{r}_2]) \leftrightarrow [\rho_{12}]$, the latter by condition (i) of Theorem 3. The other conditions of Theorem 3 give

$$\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = [\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

$$\varphi\Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = \varphi[\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

$$\Theta_1([\delta_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = [\rho_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

$$\varphi\Theta_1([\delta_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = \varphi[\rho_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

$\Theta_2 \circ a_2([\mathfrak{r}_2])$ is compatible with any other proposition which arises in the four equations. Trivial computation leads to

$$\Theta_1(([\epsilon_1] \wedge \varphi([\delta_1]) \vee [\delta_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$
$$= (([\mu_{12}] \wedge \varphi[\rho_{12}]) \vee [\rho_{12}]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$
$$\geq [\mu_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = \Theta_1([\epsilon_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

Application of Lemma 3 leads to

$$([\epsilon_1] \wedge \varphi[\delta_1]) \vee [\delta_1] \geq [\epsilon_1]$$

which is equivalent to $[\epsilon_1] \leftrightarrow [\delta_1]$ (cf. Piron, 1964, Appendice II, Theorème VII). This proves the first statement of the theorem.

Let $[\sigma_{12}]$ denote any element of the sublattice of $[\mathscr{Q}_{12}]$ generated by $\{[\mu_{12}], [\rho_{12}]\}$, and let $[\alpha_1]$ denote the corresponding, i.e. in the same way defined, element of the sublattice of $[\mathscr{Q}_1]$ generated by $\{[\epsilon_1], [\delta_1]\}$. One derives in the same manner as above

$$\Theta_1([\alpha_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = [\sigma_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

$$\varphi\Theta_1([\alpha_1]) \wedge \Theta_2 \circ a_2([\mathfrak{r}_2]) = \varphi[\sigma_{12}] \wedge \Theta_2 \circ a_2([\mathfrak{r}_2])$$

Compatibility of $[\sigma_{12}]$ with $\Theta_2 \circ a_2([\mathfrak{r}_2])$ is trivial. So conditions (i) through (iii) of Theorem 3 are fulfilled which proves $[\nu\alpha_1] = \varphi[\alpha_1]$. This completes the proof of Theorem 4.

A straightforward generalisation would be: Let us be given a boolean sublattice of $[\mathscr{Q}_{12}]$ such that for a generating set of propositions conditions (i) through (iii) of Theorem 3 are fulfilled. Then the corresponding propositions in $[\mathscr{Q}_1]$ defined by the measurement process form a boolean sublattice. In other words, we have a sufficient condition for an observable of the compound system to define an observable of the quantum object.

Another interesting problem is whether compatible propositions are always commensurable. This can be shown in Hilbert-space formulation of the measurement process (Hellwig, 1969). Here we have dropped the covering law, i.e. we are working with weaker presumptions than in Hilbert space. Until now we have not succeeded in showing that compatibility implies commensurability.

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